How many colours does Chromagon need?

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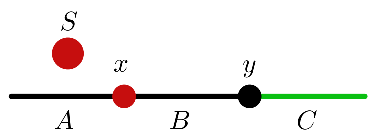
Figure 1: A 1D maze requiring 3 colours



1 What is Chromagon?

A maze consists of some coloured regions separated by coloured dividers. A coloured player marker starts in one of the regions. The player marker may move to an adjacent region cross any line whose colour matches its own. When the marker moves it takes on the colour of the region that it was in at the start of the move.

See `http://foolswood.github.io/Hexagongame/` for an interactive version of these rules applied to tiled hexagons.

1.1 What is a necessary colour?

Mazes are considered equivalent if all their spatial paths are the same. A trivial example of equivalent mazes would be if all the regions and lines (elements) of one colour were to swap colour with all the elements of another.

1.2 1D (straight line) mazes

The simplest maze type that the rules can be applied to is one where each region borders 2 others. These are equivalent to a single row maze of any other shape.

2 Proving a maze cannot be reduced

Consider the maze shown in figure 1 where S is the starting colour, A, B and C are the colours of the corresponding regions and x and y are the colours of the dividers.

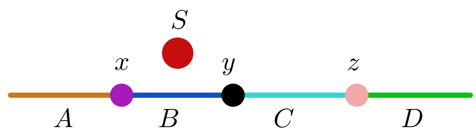
This maze can be expressed in terms of the set of equalities and inequalities that are required to impose the required movement restrictions:

$$S = x \tag{1}$$

$$A \neq x \tag{2}$$

$$A = y \tag{3}$$

Figure 2: A section of a 1D maze showing every region that can interact with y



$$B = y \tag{4}$$

$$C \neq y \tag{5}$$

$$C \neq x \tag{6}$$

Substituting equation 4 into 2 yields:

$$y \neq x \tag{7}$$

which when considered with equations 5 and 6 can be seen to require at least 3 colours to solve.

3 The maximum number of colours required for any maze

The number of colours required by a given maze is determined by the set of inequalities that restrict the available paths. Any equalities will force colour reuse (and therefore cannot increase the total number of required colours).

For instance to require 2 colours requires $a \neq b$, but to require 3 (as in section 2) requires a set of inequalities of the form:

$$a \neq b \quad \text{and} \quad b \neq c \quad \text{and} \quad c \neq a \tag{8}$$

Note that in order to have a set of equations like this it is necessary to have 2 things that are simultaneously $\neq a$.

More generally, in order to require a n colours a single element must be able to participate simultaneously in $n-1$ inequalities.

3.1 1D

Consider the interactions of y in figure 2.

For C and D to have any effect at y the path must turn around at D. In order for this to be possible:

$$B = C \tag{9}$$

which implies that the possible inequalities $B \neq y$ and $C \neq y$ are redundant.

If $A \neq y$ and $S \neq y$ then y cannot be traversed and thus no inequalities referring to B, C or D can take effect.

If $S \neq y$ then to traverse y and bring the remaining potential inequalities ($B \neq y$ and $D \neq y$) into play requires the path to return to B which requires $x = B = S$ making $B \neq y$ and $S \neq y$ redundant.

In order to return to the line from A, and thus apply $A \neq y$ requires that $x = B = S$, again this makes the constraint redundant.

Therefore the maximum number of simultaneous inequalities that can be affect y in any maze is 2, and thus 3 colours is the maximum required to express any 1D maze.

3.2 A more general upper bound

A maze of n sided shapes can be thought of as a set of regions (shapes) with connections to at most n other regions (lines). The number of possible colours we can be in a region cannot be more than the number of connections to it (plus 1 for the region in which the traversal starts).

Consider a connection between 2 regions. For the colours of the regions at either end of the connection to themselves contribute inequalities with that connection, it is necessary to be able to pass through the connection so their colour can take effect in an adjoining region. Therefore one of the surrounding regions (or the start colour) must match the colour of the connection, which means that both cannot simultaneously contribute effectual inequalities.

Thus an upper bound on the number of colours needed to express an n sided maze is given by:

$$2(n-1)+1 \tag{10}$$